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QUARTIC STRUCTURES ON SPHERES

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1. Introduction

A C^∞ tensor field f of type $(1, 1)$ on a connected C^∞ manifold P is said to define a polynomial structure of degree d if d is the smallest integer for which the powers $I, f, \cdots, f^d$ are dependent, and f has constant rank on P, where I is the identity transformation field. An almost complex manifold is a polynomial structure of degree 2. In the odd dimensional case, the almost contact manifolds provide examples of polynomial structures of degree 3. More generally, a globally framed f-manifold is a polynomial structure of degree 3. These are almost product spaces. In addition, almost product spaces are a source of further examples of polynomial structures [8].

The affine spaces R^{2n} and R^{2n-1} may be endowed with almost complex and almost contact structures, respectively, so these give the simplest examples of the manifolds considered the former having rank $2n$ and the latter rank $2n-2$. On the other hand, an odd dimensional sphere S^{2n-1} carries an almost contact structure, so it is a polynomial manifold which is globally framed. However, the even dimensional spheres are not almost complex except in dimensions 2 and 6, and whereas the contact structure on S^{2n-1} is "integrable", it is not even known whether S^6 can be given an almost complex structure which comes from a complex structure.

In a previous work [6], polynomial structures f of degree 4 were introduced and examples of them provided. These were of two types, namely,

$$f^4 + f^2 = 0 , \qquad (f^2 + I)^2 = 0 ,$$

the first one having rank $2n - 1$ and the second maximal rank $2n$. Moreover, the former is globally framed and the latter is not. We show below that, except for a set of measure zero, the even dimensional spheres may be endowed with a quartic structure f, depending on a parameter λ, that is,

$$(f^2 + \lambda^2 I)(f^2 + I) = 0 , \qquad 0 < |\lambda| \leq 1 ,$$

and

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$$f^3 + f = 0$$

on the set of points for which $\lambda = 0$. On the other hand, each of the connected components $S^{2n}\backslash\{p \in S^{2n} \mid \lambda(p) = 0\}$ may be characterized as a complete totally umbilical noninvariant hypersurface of a cosymplectic manifold of constant curvature.

It is these structures which we are preoccupied with below, that is, with even or odd dimensional manifolds carrying a quartic structure of arbitrary rank.

2. Noninvariant hypersurfaces

We motivate our continuing study of polynomial structures by considering a $2n$-dimensional manifold P imbedded as a hypersurface in an almost contact manifold $M(\varphi, E, \eta)$, $i\colon P \to M$ being the imbedding, φ the fundamental affine collineation of M, E its fundamental vector field and η the contact form. It is assumed that E is nowhere tangent to the hypersurface $i(P)$. Denoting by i_* the induced tangent map of i, we therefore have

$$\varphi i_* X = i_* JX + \alpha(X)E\ , \qquad \varphi E = 0\ , \tag{2.1}$$

where J is an almost complex structure on P and $\alpha(JX) = \eta(i_* X)$. Thus P admits an almost complex structure J and a 1-form α whose vanishing at $p \in P$ means that the tangent hyperplane at p of the hypersurface is invariant by φ_p. If $\alpha \neq 0$, the submanifold $i(P)$ is called a *noninvariant hypersurface* [3].

It is well-known that a metric G may be introduced on M with the properties

$$G(\varphi x, y) = -G(x, \varphi y)\ ,$$

where x and y are vector fields on M, that is, φ is skew symmetric with respect to G, and

$$\eta = G(E, \cdot)\ . \tag{2.2}$$

Let N be the unit normal to $i(P)$ with respect to G. Then we may write

$$\varphi i_* X = i_* fX + \alpha'(X)N \tag{2.3}$$

for some (1,1) tensor field f and 1-form α' on P. Moreover, since φN is orthogonal to N with respect to G, it is tangent to $i(P)$ and so can be expressed as

$$\varphi N = -i_* U \tag{2.4}$$

for some vector field U on P. There is clearly a relation between f, J, α and α'. Indeed, since E is nowhere tangent to P,

$$E = i_* V + \lambda N\ , \qquad \lambda \neq 0\ , \tag{2.5}$$

for some vector field V and scalar field λ on P. Clearly, $|\lambda| \leq 1$.

From (2.2) and (2.5), $\eta(N) = \lambda$. Comparing (2.1) with (2.3) yields, by virtue of (2.5),

$$f = J + \alpha \otimes V , \qquad \alpha' = \lambda\alpha .$$

If we put $\beta = i^*\eta$ and $(C\alpha)(X) = \alpha(JX)$, then $\beta = C\alpha$. Summarizing, we have proved (cf. [1])

Theorem 1. *Let $P(J, \alpha)$ be a noninvariant hypersurface of the almost contact metric manifold $M(\varphi, \eta, G)$. Then there exist tensor fields $f, U, V, \alpha, \alpha', \beta$ and $\lambda, 0 < |\lambda| \leq 1$, on P defined by*

(i) $\varphi i_* X = i_* fX + \alpha'(X)N, \quad \alpha' = \lambda\alpha,$
(ii) $\varphi N = -i_* U,$
(iii) $E = i_* V + \lambda N,$
(iv) $\beta = C\alpha = i^*\eta,$
(v) $\lambda = \eta(N).$

The vector fields U and V vanish at those points on P for which $\lambda = \pm 1$, and at these points f coincides with J.

Noting that $\varphi^2 = -I + \eta \otimes E$, Theorem 1 yields

Proposition 2. *The structure tensors on the noninvariant hypersurface P satisfy*

(i) $f^2 = -I + \alpha' \otimes U + \beta \otimes V,$
(ii) $\alpha' \circ f = \lambda\beta, \quad \beta \circ f = -\lambda\alpha',$
(iii) $fU = -\lambda V, \quad fV = \lambda U,$
(iv) $\alpha'(U) = 1 - \lambda^2, \quad \alpha'(V) = 0,$
(v) $\beta(U) = 0, \quad \beta(V) = 1 - \lambda^2.$

Corollary 1. *$P(f, U, V, \alpha, \beta, \lambda)$ is a quartic structure of maximal rank.*

In fact,

$$f^4 + (1 + \lambda^2)f^2 + \lambda^2 I = 0 ,$$

that is,

$$(f^2 + \lambda^2 I)(f^2 + I) = 0 .$$

If $fX = 0$, then $X = \alpha'(X)U + \beta(X)V$. Hence $0 = \alpha'(X)f^2U + \beta(X)f^2V = -\lambda^2(\alpha'(X)U + \beta(X)V)$, so λ being nowhere zero, X must vanish.

In the sequel, unless explicitly mentioned to the contrary, we assume that $\lambda \neq \pm 1$ and, in this case, we denote by P' the manifold defined by $\{p \in P \mid 0 < \lambda(p) < 1\}$. Geometrically this means that E and N are distinct vector fields in the sense that $E_{i(p)} \neq N_{i(p)}$ for all $p \in P'$. This means that the Euler-Poincaré characteristic $\chi(P')$ of P' is zero since U (and V) are nonsingular vector fields on P'. Topologically, this is the case for the odd dimensional spheres, but $\chi(S^{2n}) \neq 0$. Note however that there are points on $S^{2n} - \{p \in S^{2n} \mid \lambda(p)$

$= 0\}$, considered as a noninvariant hypersurface, for which $\lambda = \pm 1$.

If we put $\tilde{\alpha} = (1-\lambda^2)^{-1}\alpha'$ and $\tilde{\beta} = (1-\lambda^2)^{-1}\beta$, then P' is *pseudo-globally framed*, that is, $\tilde{\alpha}(U) = 1, \tilde{\alpha}(V) = 0, \tilde{\beta}(U) = 0, \tilde{\beta}(V) = 1$ and

$$f^2 = -I + \rho(\tilde{\alpha} \otimes U + \tilde{\beta} \otimes V)\ , \qquad \rho = 1 - \lambda^2\ .$$

Corollary 2. *The noninvariant hypersurface P' is pseudo-globally framed. Moreover,*

$$\tilde{\alpha} \circ f = \lambda\tilde{\beta}\ , \qquad \tilde{\beta} \circ f = -\lambda\tilde{\alpha}\ ,$$

and

$$fU = -\lambda V\ , \qquad fV = \lambda U\ .$$

If we put

$$f_1 = f - \lambda\tilde{\beta} \otimes U$$

and compute its square, we get $f_1^2X = -X + \tilde{\alpha}(X)U + \tilde{\beta}(X)V$. Moreover, $f_1U = -\lambda V$ and $f_1V = 0$. Hence $f_1^3X = -f_1X - \lambda\tilde{\alpha}(X)V$, from which $f_1^4X = -f_1^2X$. If $f_1X = 0$, then $X = \tilde{\alpha}(X)U + \tilde{\beta}(X)V$. Applying f_1 again gives $0 = -\lambda\tilde{\alpha}(X)V$, so $\tilde{\alpha}(X) = 0$ since λ and V are nowhere zero. Thus, rank $f_1 = 2n-1$.

An even dimensional C^∞ manifold P is said to be *globally framed* if there exist a linear transformation field f, global vector fields E_a and 1-forms η^a, $a = 1, \cdots, 2\nu$, satis fying

$$\eta^a(E_b) = \delta^a_b\ ,$$
$$f^2 = -I + \eta^a \otimes E_a\ ,$$

the summation convention being employed here and in the sequel.

Theorem 3. *The noninvariant hypersurface P' carries a globally framed quartic structure $(f_1, U, V, \tilde{\alpha}, \tilde{\beta})$ of rank $2n-1$, where $f_1 = f - \lambda\tilde{\beta} \otimes U$, that is,*

(i) $f_1^2 = -I + \tilde{\alpha} \otimes U + \tilde{\beta} \otimes V$,

(ii) $\tilde{\alpha} \circ f_1 = 0, \quad \tilde{\beta} \circ f_1 = -\lambda\tilde{\alpha}$,

(iii) $f_1U = -\lambda V, \quad f_1V = 0,$

(iv) $\tilde{\alpha}(U) = 1, \quad \tilde{\alpha}(V) = 0$,

(v) $\tilde{\beta}(U) = 0, \quad \tilde{\beta}(V) = 1$,

and

$$f_1^4 + f_1^2 = 0\ .$$

Corollary. *The noninvariant hypersurface $P'(J, \alpha, g)$ of the almost contact metric manifold $M(\varphi, \eta, G)$ may be endowed with a quartic structure of maximal rank $2n$ given by the $(1,1)$ tensor field*

$$J - \lambda\left(\alpha \otimes JU + \frac{1}{1-\lambda^2} C\alpha \otimes U\right),$$

where U is the vector field on P' (of length 1) given by $\varphi N = -i_* U$ and the function λ is defined by $E = \lambda(N - i_* JU), \eta = G(E, \cdot)$.

If we put $f_2 = f_1 + \lambda\tilde{\alpha} \otimes V$, then $f_2 U = 0$ and $f_2 V = 0$. Moreover, $f_2^2 X = -X + \tilde{\alpha}(X)U + \tilde{\beta}(X)V$. Applying f_2 again gives $f_2^3 X = -f_2 X$.

Theorem 4. *The noninvariant hypersurface P' carries a globally framed f-structure $(f_2, U, V, \alpha, \beta)$ of rank $2n-2$, that is,*

(i) $f_2^2 = -I + \tilde{\alpha} \otimes U + \tilde{\beta} \otimes V$,
(ii) $\tilde{\alpha} \circ f_2 = 0, \quad \tilde{\beta} \circ f_2 = 0$,
(iii) $f_2 U = 0, \quad f_2 V = 0$,
(iv) $\tilde{\alpha}(U) = 1, \quad \tilde{\alpha}(V) = 0$,
(v) $\tilde{\beta}(U) = 0, \quad \tilde{\beta}(V) = 1$,

where

$$f_2 = f - \lambda(\tilde{\beta} \otimes U - \tilde{\alpha} \otimes V) .$$

Corollary. *The noninvariant hypersurface $P'(J, \alpha, g)$ of the almost contact metric manifold $M(\varphi, \eta, G)$ may be endowed with an f-structure of rank $2n-2$ defined by the $(1,1)$ tensor field*

$$J - \frac{\lambda}{1-\lambda^2}(C\alpha \otimes U + \alpha \otimes JU),$$

where U is the vector field on P (of length 1) given by $\varphi N = -i_* U$ and the scalar field λ is given by $E = \lambda(N - i_* JU), \eta = G(E, \cdot)$.

3. Symplectic quartic structures

Since the collineation φ is skew symmetric with respect to the metric G, the field f has this property with respect to the induced metric. For, $g(fX, Y) = G(i_* fX, i_* Y) = G(\varphi i_* X - \alpha'(X)N, i_* Y) = -G(i_* X, \varphi i_* Y) = -G(i_* X, i_* fY) = -g(X, fY)$. We put

$$F(X, Y) = g(fX, Y)$$

and call F the *fundamental form* of the noninvariant hypersurface $P(f, U, V, g)$. Clearly,

$$F = i^* \Phi ,$$

where Φ is the fundamental form of the ambient space.

From (2.3)–(2.5),

$$\alpha' = g(U, \cdot)\ , \qquad \beta = g(V, \cdot)\ .$$

Moreover, from formula (iv) of Proposition 2, U and V are orthogonal, that is, $g(U, V) = 0$. If we put $\bar{g} = (1-\lambda^2)^{-1}g$ on P', then U and V form an orthonormal pair with respect to the metric $\bar{g}$. Setting $F_2 = g(f_2X, Y)$, we get

$$F_2 = F + 2\lambda(1 - \lambda^2)\tilde{\alpha} \wedge \tilde{\beta}\ .$$

Assume now that M is normal and Φ is closed; for example, assume that M is a cosymplectic manifold. Then, F is closed, so by Corollary 1 to Proposition 2, the 'quartic' structure on P is symplectic, that is, F is closed and of maximal rank. (The ambient space cannot be a contact metric manifold since P is a noninvariant hypersurface [3].) Observe that $F(X, Y) = g(fX, Y) = g(JX, Y) + \alpha(X)\beta(Y)$, so $F(X, Y) = \frac{1}{2}[g(JX, Y) - g(X, JY)] + (\alpha \wedge \beta)(X, Y)$. Since $\alpha \neq 0$, (J, g) is not an hermitian structure on P. However, (J, g^*) is hermitian where $g^* = g + \alpha \otimes \alpha$. Indeed, $g^*(JX, Y) = g(JX, Y) + \beta(X)\alpha(Y) = F(X, Y) - \alpha(X)\beta(Y) + \alpha(Y)\beta(X) = F(X, Y) - 2(\alpha \wedge \beta)(X, Y)$. Putting

$$\Omega^*(X, Y) = g^*(JX, Y)\ ,$$

it is easily seen that

$$F = \Omega^* + 2\alpha \wedge \beta\ .$$

Since f is of maximal rank and i is a regular map, the symmetric tensor

$$\gamma = g - \beta \otimes \beta$$

defines a Riemannian metric, in fact, an hermitian metric with respect to J. Since Φ is closed, γ is an almost Kaehler metric and F is the fundamental form of the almost Kaehler manifold $P(J, \gamma)$. M being normal, $P(J, \gamma)$ is a Kaehler manifold [3].

Denote by K the ring of real-valued functions on P. To each vector field X on P, we assign the 1-form ξ defined by

$$\xi = \iota(X)F\ ,$$

where ι is the interior product operator given by

$$[\iota(X)\theta](X_1, \cdots, X_{p-1}) = p\theta(X, X_1, \cdots, X_{p-1})\ ,$$

θ being a p-form, $p \geq 1$, and $\iota(X)k = 0$, $k \in K$. We therefore have an isomorphism μ of the K-module of vector fields on P onto the K-module of 1-forms on P defined by $\mu(X) = \xi$. This isomorphism may be naturally extended to an isomorphism, again denoted by μ, of the K-module of skew symmetric contravariant tensors of order p with the K-module of p-forms.

Following P. Libermann (see [7]) an operator $\tilde{*}$ analogous to the Hodge star operator is defined as follows:

$$\tilde{*}\theta = \iota(\mu^{-1}\theta)\frac{F^n}{n!} .$$

If θ is a p-form, then $\tilde{*}\theta$ is a $(2n-p)$-form and

$$\tilde{*}^2\theta = \theta .$$

We may also define the operator $L = \epsilon(F)$, the exterior product by F, on the symplectic manifold $P(f, g)$. It clearly coincides with the corresponding operator of Hodge-Weil on Kaehler manifolds. We shall see below that its "dual" $\tilde{\Lambda}$ with respect to $\tilde{*}$ coincides with its dual with respect to $*$. In fact,

$$\tilde{\Lambda} = \tilde{*}^{-1}L\tilde{*} = \tilde{*}L\tilde{*} . \tag{3.1}$$

Thus

$$\tilde{\Lambda}\theta = \iota(\mu^{-1}F)\theta , \qquad p \geq 2 .$$

Analogous to the codifferential operator δ, the *symplectic codifferential operator* $\tilde{\delta}$ is defined by

$$\tilde{\delta} = \tilde{*}^{-1}d\tilde{*} = \tilde{*}d\tilde{*} . \tag{3.2}$$

Clearly

$$\tilde{\delta}^2 = 0 .$$

We relate the operators $\tilde{*}$, $\tilde{\Lambda}$ and $\tilde{\delta}$ on the symplectic manifold P to the corresponding operators of Hodge-Weil on the underlying almost hermitian structure (J, γ). Since $F(X, Y) = \gamma(JX, Y)$, we obtain

$$\frac{F^n}{n!} = (-1)^{n(n-1)/2} * 1 , \tag{3.3}$$

where $*1$ is the volume element of γ. Clearly,

$$*^2 = (-1)^p$$

on p-forms.

The operator C previously applied to α may be extended to a mapping, again denoted by C, on p-forms θ as follows:

$$C\theta(X_1, \cdots, X_p) = \theta(JX_1, \cdots, JX_p) .$$

It has the obvious property

$$C^2\theta = (-1)^p\theta \ .$$

Moreover, it commutes with $*$ and L. From the definition of $\tilde{*}$, we obtain

$$\tilde{*} = (-1)^{n(n-1)/2} * C^{-1} \ .$$

Hence

$$\tilde{\Lambda}\theta = \tilde{*}^{-1}L\tilde{*}\theta = C*^{-1}L*C^{-1}\theta = C\Lambda C^{-1}\theta = CC^{-1}\Lambda\theta = \Lambda\theta \ ,$$

so the operators Λ and $\tilde{\Lambda}$ coincide.

From (3.2), we get

$$\tilde{\delta} = C\delta C^{-1} \ . \tag{3.4}$$

For, $\tilde{\delta}\theta = C*^{-1}d*C^{-1}\theta = C\delta C^{-1}\theta$. Hence, from the formula

$$d\Lambda - \Lambda d = -C\delta C^{-1}$$

valid for almost hermitian manifolds, the purely symplectic relations

$$d\Lambda - \Lambda d = -\tilde{\delta} \ ,$$

and

$$\tilde{\delta}L - L\tilde{\delta} = -d$$

are obtained.

Finally, from (3.3) and (3.4),

$$\int_P \tilde{\delta}\theta\tilde{*}1 = 0$$

if P is compact.

If P' is compact, its topology can therefore be studied from the symplectic point of view (f, g) in comparison with the symplectic point of view (J, γ) of Hodge-Weil, the interesting thing being that F is the fundamental from of both structures.

4. Hypersurfaces of cosymplectic spaces

Denote by ∇ the Riemannian connection of $M(\varphi, \eta, G)$ and by D the induced connection on the hypersurface P. Then, the equations of Gauss and Weingarten are

$$(D_X i_*)Y = h(X, Y)N$$

and

$$D_X N \equiv \nabla_{i_* X} N = -i_* HX$$

respectively, where h and H are the second fundamental tensors (of types (0,2) and (1,1), respectively) of P with respect to the normal vector field N. The tensor h is symmetric and $h(X, Y) = g(HX, Y)$.

Covariant differentiation of (2.3) along P yields

$$\begin{aligned}(\nabla_{i_* Y}\varphi)i_* X - h(X, Y)i_* U + i_* f D_Y X + \alpha'(D_Y X)N \\ = h(fX, Y)N + i_*(D_Y f)X + i_* f D_Y X + (D_Y \alpha')(X)N \\ + \alpha'(D_Y X)N - \alpha'(X)i_* HY .\end{aligned}$$

Covariant differentiation of (2.4) along P gives

$$(\nabla_{i_* X}\varphi)N - \varphi i_* HX = -h(X, U)N - i_* D_X U ,$$

from which follows

$$(\nabla_{i_* X}\varphi)N - i_* fHX - \alpha'(HX)N = -h(X, U)N - i_* D_X U .$$

Differentiating (2.5) gives rise to

$$\nabla_{i_* X} E = h(X, V)N + i_* D_X V + (X\lambda)N - \lambda i_* HX .$$

Similarly we have, from $\beta = i^*\eta$,

$$(D_X\beta)(Y) + \beta(D_X Y) = (\nabla_{i_* X}\eta)(i_* X) + h(X, Y)\eta(N) + \beta(D_X Y) ,$$

and, from $\lambda = \eta(N)$,

$$X\lambda = (\nabla_{i_* X}\eta)(N) - \beta(HX) .$$

If the ambient space is cosymplectic, then

$$\nabla\varphi = 0 , \qquad \nabla\eta = 0 , \qquad \nabla E = 0$$

(see [3]). Thus

$$\begin{aligned}&(D_X f)Y = \alpha'(Y)HX - h(X, Y)U , \\ &(D_X\alpha')(Y) = -h(X, fY) , \qquad (D_X\beta)(Y) = \lambda h(X, Y) , \\ &D_X U = fHX , \qquad D_X V = \lambda HX , \\ &h(X, U) = \alpha'(HX) , \qquad h(X, V) = \beta(HX) , \\ &X\lambda = -\beta(HX) .\end{aligned} \tag{4.1}$$

Observe that β is closed.

The manifold P' is said to be *totally flat* if its structure tensors are parallel fields with respect to the Riemannian connection.

P' is called *normal* if its induced globally framed f-structure is normal [4].

Proposition 5. *If the ambient space of the symplectic quartic manifold P is cosymplectic and $h = \mu\alpha \otimes \alpha$, then f is covariant constant and $\lambda =$ const. On the other hand, if P is totally umbilical, then V is a conformal vector field. If H commutes with f, then U is a Killing vector field, and conversely.*

Although f has vanishing covariant derivative, (f, g) is not normal (see § 6). However, we do have

Corollary 1. *If P is totally geodesic and the ambient space is cosymplectic, then P is totally flat and normal.*

If P satisfies the conditions in Corollary 1, then $DJ = 0$ since $J = f - \alpha \otimes V$. Hence (J, g^*), as well as (J, γ), is a Kaehler structure on P.

Corollary 2. *The hermitian structure (J, g^*) on the totally geodesic hypersurface P of the cosymplectic manifold M is Kaehlerian.*

Corollary 3. *If the hypersurface (P, g) is totally umbilical, that is, if $h = \sigma g$ (with respect to N) and (M, G) is of constant curvature, then for $n > 1, (P, g)$ is of constant nonnegative curvature σ^2.*

Proof. Indeed, $\nabla_{i_* X} i_* Y = i_* D_X Y + \sigma g(X, Y)N$, so

$$\nabla_{i_* Z}\nabla_{i_* X} i_* Y = i_* D_Z D_X Y + \sigma\{g(Z, D_X Y) + g(D_Z X, Y) + g(X, D_Z Y) + Z\sigma \cdot g(X, Y)\}N - \sigma^2 g(X, Y) i_* Z \ .$$

Denoting by $\tilde{R}$ and R the curvature tensors of the metrics G and g, respectively, we get

$$\tilde{R}(i_* Z, i_* X) i_* Y = i_*\{R(Z, X)Y - \sigma^2[g(X, Y)Z - g(Z, Y)X]\} + \{Z\sigma \cdot g(X, Y) - X\sigma \cdot g(Z, Y)\}N \ .$$

Thus, $\tilde{R}$ being zero,

$$R(Z, X)Y = \sigma^2\{g(X, Y)Z - g(Z, Y)X\}$$

and $Z\sigma \cdot g(X, Y) = X\sigma \cdot g(Z, Y)$, each of these relations implying σ is constant on M.

Under the conditions of Corollary 3, equations (4.1) become

$$(4.2) \quad \begin{aligned} &(D_X f)Y = \sigma[\alpha'(Y)X - g(X, Y)U] \ , \\ &(D_X \alpha')(Y) = \sigma F(X, Y) \ , \qquad (D_X \beta)(Y) = \lambda\sigma g(X, Y) \ , \\ &D_X U = \sigma f X \ , \qquad D_X V = \sigma\lambda X \ , \\ &X\lambda = -\sigma\beta(X) \ . \end{aligned}$$

Observe that $F = -D\alpha'$, so F is an exact 2-form, that is, F does not give rise to a nontrivial cohomology class. Moreover, the second and third statements of Proposition 5 clearly hold. It may also be checked that $d\alpha$ is of bidegree (1,1) with respect to $J = f - \alpha \otimes V$.

In [3] it was shown that a noninvariant hypersurface of a cosymplectic manifold is a complex manifold. Thus $P(J, g)$, as well as $P'(J, g)$, is endowed with the integrable almost complex structure of formula (2.1).

The even dimensional sphere: We regard E^{2n+1} as a cosymplectic space and let $\tilde{S}^{2n}$ be the unit sphere in E^{2n+1} minus the set of points (the equator) on which $\lambda = 0$ (see also [1]). Then, since S^{2n} is a totally umbilical hypersurface of E^{2n+1} with $\sigma = -1$, equations (4.2) become

$$\begin{aligned} &(D_X f)Y = g(X, Y)U - \alpha'(Y)X \ , \\ &(D_X\alpha')(Y) = -F(X, Y) \ , \qquad (D_X\beta)(Y) = -\lambda g(X, Y) \ , \\ &D_X U = -fX \ , \qquad D_X V = -\lambda X \ , \\ &X\lambda = \beta(X) \ . \end{aligned}$$

Thus, for every n, $\tilde{S}^{2n}$ (as well as $(S^{2n})' = S^{2n}\backslash\{p \in S^{2n} \mid \lambda(p) = 0, +1, -1\}$) carries the integrable almost complex structure of formula (2.1). (Observe that none of the points on S^{2n} for which $\lambda = 0$ can be a zero of the closed conformal vector field V; see [10, p. 170].)

Corollary 4. *If the hypersurface (P, g) is totally geodesic and its ambient space (M, G) is of constant curvature, then (P, g) and (P, γ) (as well as (M, G)) are locally flat.*

Proof. Let $\bar{D}$ denote the Riemannian connection of γ. Then, by the definition of the Riemannian connection and the formulas $DJ = 0$ and $(D_X\alpha)(Y) = -h(X, JY)$(see [3]), we get

$$\begin{aligned} \gamma(\bar{D}_X Y, Z) &= \gamma(D_X Y, Z) + h(X, Y)\beta(Z) \\ &= \gamma(D_X Y, Z) - h(X, Y)\gamma(JU, Z) \end{aligned}$$

so that

$$\bar{D}_X Y = D_X Y - h(X, Y)JU = D_X Y \ , \tag{4.3}$$

since (P, γ) is a totally geodesic hypersurface [6].

Again, since $h = 0$, the Gauss equation (with respect to N) is

$$\nabla_{i_*X} i_* Y = i_* D_X Y = i_* \bar{D}_X Y \ .$$

Thus, denoting the Riemannian curvature tensor of (P, γ) by $\bar{R}$,

$$\tilde{R}(i_*X, i_*Y)i_*Z = [\nabla_{i_*X}, \nabla_{i_*Y}]i_*Z - \nabla_{[i_*X, i_*Y]}i_*Z = i_*\bar{R}(X, Y)Z \ ,$$

from which γ is a locally flat metric since $\tilde{R} = 0$ and i is a regular map.

Corollary 5. *Under the conditions of Corollary* 3, *if* $P(J, g)$ *is an invariant hypersurface or* $\lambda = \pm 1$, *then* P *is loocally flat* $(n > 1)$.

The is an immediate consequence of the formula

$$\bar{R}(X, Y)Z$$
$$= R(X, Y)Z + \sigma^2\left\{g(Y, Z)\left(\frac{1}{\lambda^2}\beta \otimes V + I\right)X - g(Z, X)\left(\frac{1}{\lambda^2}\beta \otimes V + I\right)Y\right\}$$

obtained from (4.3) by putting $h = \sigma g$.

Remark. Since in the totally flat case, λ is constant, we may consider the one parameter family of hypersurfaces P_λ, $0 < |\lambda| < 1$. For each value of λ, P_λ is a noninvariant hypersurface carrying a globally framed quartic structure f_λ of rank $2n - 1$, that is,

$$f_\lambda^2 = -I + \tilde{\alpha} \otimes U + \tilde{\beta} \otimes V ,$$
$$f_\lambda U = -\lambda V , \qquad f_\lambda V = 0 ,$$
$$\tilde{\alpha}(U) = 1 , \qquad \tilde{\alpha}(V) = 0 ,$$
$$\tilde{\beta}(U) = 0 , \qquad \tilde{\beta}(V) = 1 .$$

P_λ also carries a globally framed f-structure f^λ of rank $2n - 2$ where

$$f^\lambda = f - \lambda(\tilde{\beta} \otimes U - \tilde{\alpha} \otimes V) .$$

There are no noninvariant hypersurfaces of a normal contact metric manifold [3].

5. Metric polynomial manifolds

An m-dimensional C^∞ manifold P is said to be pseudo-globally framed (see § 2) if there exist a C^∞ tensor field f of type (1,1), global vector fields E_a and 1-forms $\eta^a, a = 0, 1, \cdots, 2\nu$, with $E_0 = 0$, $\eta^0 = 0$, satisfying

$$\eta^a(E_b) = \delta^a_b , \tag{5.1}$$
$$f^2 = -I + (1 - \lambda^2)\eta^a \otimes E_a , \qquad |\lambda| \leq 1 . \tag{5.2}$$

If $\lambda = 0$, then (f, E_a, η^a) is *globally framed* (see [4]). If $\lambda = \pm 1$, then f defines an almost complex structure, so m is even. If, in addition, $m = 2\nu$, P is a parallelizable almost complex manifold. (Observe that a compact complex parallelizable manifold is Kaehlerian, if and only if, it is a multi-torus [9].) Clearly, the only pseudo-globally framed polynomial structures defined by f are those given by $f^2 + I = 0, f^3 + f = 0$ and $f^4 + (1 + \lambda^2)f^2 + \lambda^2 I = 0, \lambda \not\equiv \pm 1$, the second case arising by assuming $fE_a = 0, a = 1, \cdots, 2\nu$, and the latter including $f^4 + f^2 = 0$ as a special case. In the sequel, we study those structures for which $\lambda \neq 0$. The case $\lambda \equiv 0$ was the subject of a previous paper [6].

From (5.2), $f^2 E_a = -\lambda^2 E_a$, $a = 1, \cdots, 2\nu$.

If $fX = 0$, then $X = (1 - \lambda^2)\eta^a(X)E_a$. Applying f^2 to this relation yields $\lambda^2(1 - \lambda^2)\eta^a(X)E_a = 0$, so $X = 0$ at those points for which $\lambda \neq \pm 1$, since $\lambda \neq 0$. At those $p \in P$ where $\lambda(p) = \pm 1, f_p^2 = -I_p$. (We shall presently see that the dimension of P is even.) Hence the linear transformation field f has maximal rank everywhere on P.

We assume below that

$$fE_{2i-1} = -\lambda E_{2i} , \tag{5.3}$$

$$fE_{2i} = \lambda E_{2i-1} , \qquad i = 1, \cdots, \nu . \tag{5.4}$$

The pseudo-globally framed manifold $P(f, E_a, \eta^a)$ is called a *pseudo-globally framed metric manifold* if it carries a Riemannian metric g such that

(i) $\eta^a = g(E_a, \cdot)$,

(ii) $g(fX, Y) = -g(X, fY)$.

We put $F(X, Y) = g(fX, Y)$ and call F the *fundamental form* of $P(f, \eta^a, g)$. From (i) and (ii), we obtain

$$\eta^{2i-1} \circ f = \lambda\eta^{2i} , \tag{5.5}$$

$$\eta^{2i} \circ f = -\lambda\eta^{2i-1} . \tag{5.6}$$

If we put

$$J = f - \frac{1}{\lambda}\eta^{2i-1} \otimes E_{2i} , \tag{5.7}$$

then J is an almost complex structure, so $m = 2n$.

Theorem 6. *The manifold P with structure tensors (f_1, η^a, g), $a = 1, \cdots, 2\nu$, where $f_1 = f - \lambda\eta^{2i} \otimes E_{2i-1}$, $i = 1, \cdots, \nu$, is an even dimensional globally framed quartic structure of rank $m - \nu$, that is,*

(i) $f_1^2 = -I + \eta^a \otimes E_a$,

(ii) $\eta^{2i-1} \circ f_1 = 0, \quad \eta^{2i} \circ f = -\lambda\eta^{2i-1}$,

(iii) $f_1 E_{2i-1} = -\lambda E_{2i}, \quad f_1 E_{2i} = 0$,

(iv) $\eta^a(E_b) = \delta_b^a$,

(v) $f_1^4 + f_1^2 = 0$.

Proof. Employing (5.3)–(5.6),

$$\begin{aligned} f_1^2 X &= (f - \lambda\eta^{2i} \otimes E_{2i-1})(fX - \lambda\eta^{2j}(X)E_{2j-1}) \\ &= f^2 X + \lambda^2\eta^{2j}(X)E_{2j} + \lambda^2\eta^{2i-1}(X)E_{2i-1} \\ &= -X + (1 - \lambda^2)\eta^a(X)E_a + \lambda^2\eta^a(X)E_a \\ &= -X + \eta^a(X)E_a . \end{aligned}$$

Moreover, $f_1 E_{2i-1} = -\lambda E_{2i}, f_1 E_{2i} = \lambda E_{2i-1} - \lambda E_{2i-1} = 0$. Hence $f_1^3 X = -f_1 X - \lambda\eta^{2i-1}(X)E_{2i}$, from which $f_1^4 X = -f_1^2 X$.

If $f_1 X = 0$, then $X = \eta^a(X)E_a$. Applying f_1 again yields $0 = -\lambda\eta^{2i-1}(X)E_{2i}$, that is, $\eta^{2i-1}(X) = 0$, so $X = \eta^{2i}(X)E_{2i}$.

Theorem 7. *The manifold P with structure tensors (f_2, η^a, g), $a = 1, \cdots, 2\nu$, where $f_2 = f - \lambda(\eta^{2i} \otimes E_{2i-1} - \eta^{2i-1} \otimes E_{2i})$, is an even dimensional globally framed f-structure of rank $m - 2\nu$, that is,*

(i) $f_2^2 = -I + \eta^a \otimes E_a$,

(ii) $\eta^a \circ f_2 = 0, \quad f_2 E_a = 0$,

(iii) $\eta^a(E_b) = \delta_b^a$.

Proof. Since $f_2 = f_1 + \lambda\eta^{2i-1} \otimes E_{2i}$, $f_2 E_{2i-1} = -\lambda E_{2i} + \lambda E_{2i} = 0$ and $f_2 E_{2i} = 0$. Hence

$$\begin{aligned} f_2^2 X &= (f_1 + \lambda\eta^{2i-1} \otimes E_{2i})(f_1 X + \lambda\eta^{2j-1}(X)E_{2j}) \\ &= f_1^2 X = -X + \eta^a(X)E_a \,. \end{aligned}$$

Moreover, $f_2^3 X = -f_2 X + \eta^a(X) f_2 E_a = -f_2 X$.

If $f_2 X = 0$, then $X = \eta^a(X)E_a$, so rank $f_2 = m - 2\nu$ since $f_2 E_a = 0, a = 1, \cdots, 2\nu$.

A pseudo-globally framed metric manifold is said to be *totally flat* if the covariant derivatives (with respect to the Riemannian connection) of its structure tensors are zero.

Theorem 8. *Let $P_\lambda(f, \eta^a, g)$ be a complete totally flat pseudo-globally framed manifold. If P_λ is simply connected, then there is a Kaehlerian submanifold whose dimension is rank f.*

Proof. Clearly, f_2 is also a parallel field, so DF_2 also vanishes where $F_2(X, Y) = g(f_2 X, Y)$. Thus $P'_{\lambda p} = \{X \in P_{\lambda p} \mid F_2(X, P_{\lambda p}) = 0\}$ defines a parallel distribution, and therefore the orthogonal complement $P''_{\lambda p}$ (with respect to g) also gives a parallel distribution. Observe that the E_a do not belong to $P''_{\lambda p}$. By the de Rham decomposition theorem $P_\lambda = P'_\lambda + P''_\lambda$ where F_2 vanishes on P'_λ and F_2 has maximal rank on P''_λ. Since Df_2 vanishes so does the Nijenhuis torsion $[f_2, f_2]$. The almost complex structure on P''_λ obtained by restricting f_2 to P''_λ is therefore integrable. F''_2 being closed, P''_λ is symplectic. In fact, since F_2 has vanishing covariant derivative, P''_λ is a Kaehler manifold.

On a pseudo-globally framed quartic metric manifold, if we define

$$\bar{f} = f + (1 - \lambda)[\eta^{2i} \otimes E_{2i-1} - \eta^{2i-1} \otimes E_{2i}]$$

and

$$\hat{f} = f - (1 + \lambda)[\eta^{2i} \otimes E_{2i-1} - \eta^{2i-1} \otimes E_{2i}] \,,$$

$i = 1, \cdots, \nu$, then $\bar{f}$ and $\hat{f}$ are almost complex structures other than J.

Moreover, $(\tilde{f}, g)$ is an almost hermitian structure on P. Setting $\tilde{F}(X, Y) = g(\tilde{f}X, Y)$, we obtain

$$\tilde{F} = F + 2(1 - \lambda) \sum_i \eta^{2i} \wedge \eta^{2i-1} .$$

If the fundamental form F and the η^a are closed forms, the almost hermitian structure on P is almost Kaehlerian. It is Kaehlerian, if and only if $\tilde{f}$ has vanishing covariant derivative with respect to g, that is, if $P(f, \eta^a, g)$ is totally flat. In this case, the E_a are infinitesimal automorphisms of the Kaehlerian structure. The same conclusion prevails if we consider $\hat{f}$ instead of $\tilde{f}$.

Theorem 9. *If the pseudo-globally framed metric manifold $P(f, \eta^a, g)$ is totally flat, then it carries the Kaehler structures $(\tilde{f}, g)$ and $(\hat{f}, g)$.*

A corresponding theory may be developed for odd dimensional manifolds by letting $a \in \{1, \cdots, 2\nu + 1\}, i \in \{1, \cdots, \nu\}$ and by setting $fE_{2\nu+1} = 0$, in which case from (5.7), $J^2 = -I + \eta^{2\nu+1} \otimes E_{2\nu+1}$. Hence J is the fundamental affine collineation of the almost contact manifold $P(J, E_{2\nu+1}, \eta^{2\nu+1})$. We give the analogues of Theorems 8 and 9:

Theorem 10. *A totally flat odd dimensional pseudo-globally framed metric manifold $P(f, \eta^a, g)$ may be endowed with the cosymplectic structures $(\tilde{f}, \eta, g)$ and $(\hat{f}, \eta, g)$.*

Proof. That P is almost cosymplectic is a consequence of the fact that f and the η^a are covariant constant, and $\lambda =$ const. Thus φ and η also have vanishing covariant derivatives. The normality of P follows from the vanishing of

$$(D_{\varphi X}\varphi)Y - (D_{\varphi Y}\varphi)X + \varphi(D_Y\varphi)X - \varphi(D_X\varphi)Y + \{(D_X\eta)(Y) - (D_Y\eta)(X)\}E .$$

Corollary. *Let $P_\lambda(f, \eta^a, g)$ be a complete totally flat odd dimensional pseudo-globally framed manifold. If P_λ is simply connected, then there is a Kaehlerian submanifold of dimension rank f.*

The proof is similar to that of Theorem 8.

6. Normal symplectic structures

Recall that the framed structure $(f, U, V, \alpha, \beta, \lambda)$ on P' is *normal* if the underlying globally framed f-structure $(f_2, U, V, \tilde{\alpha}, \tilde{\beta})$ is normal. The condition for this is given by the vanishing of the tensor field $[f_2, f_2] + d\tilde{\alpha} \otimes U + d\tilde{\beta} \otimes V$ of type (1,2).

The direct product of the pseudo-globally framed hypersurfaces $P_i(f_{(i)}, U_i, V_i, \tilde{\alpha}_i, \tilde{\beta}_i, \lambda_i), i = 1, 2$, also has the naturally induced almost complex structure $\hat{J}$ on $P'_1 \times P'_2$ defined by

$$\hat{J}_{(p_1, p_2)}(X_1, X_2) = (f_{(1)}X_1 - \lambda_1\tilde{\beta}_1(X_1)U_1 + \lambda_1\tilde{\alpha}_1(X_1)V_1 - \tilde{\beta}_2(X_2)V_1 - \tilde{\alpha}_2(X_2)U_1 ,$$
$$f_{(2)}X_2 - \lambda_2\tilde{\beta}_2(X_2)U_2 + \lambda_2\tilde{\alpha}_2(X_2)V_2 + \tilde{\beta}_1(X_1)V_2 + \tilde{\alpha}_1(X_1)U_2) .$$

If the P_i' are normal, then $\hat{J}$ is integrable [4]. The converse is obtained by employing $[\hat{J},\hat{J}](X_1 \times 0, Y_1 \times 0) = 0$, $[\hat{J},\hat{J}](0 \times X_2, 0 \times Y_2) = 0$, $[\hat{J},\hat{J}](0 \times X_2, Y_1 \times 0) = 0$ and $[\hat{J},\hat{J}](X_1 \times 0, 0 \times Y_2) = 0$ in the expression for $[\hat{J},\hat{J}](X_1 \times X_2, Y_1 \times Y_2)$ where $X \times Y = (X, Y)$. Define a metric on $P_1' \times P_2'$ by $g_1 + g_2$, where $g_j = i_j^* G$, $j = 1, 2$, is the metric induced on P_j' by the almost contact metric G of $M(\varphi, \eta, G)$. Assuming that M is cosymplectic, the 2-form $\tilde{\Omega}$ on $P_1' \times P_2'$ defined by $\tilde{\Omega} = (F_{(1)}, 0) + (0, F_{(2)}) + (\tilde{\alpha}_1, 0) \wedge (0, \tilde{\alpha}_2) + (\tilde{\beta}_1, 0) \wedge (0, \tilde{\beta}_2)$, where $F_{(j)} = i_j^* \Phi$, $j = 1, 2$, is the fundamental form of P_j', is the Kaehler form of $P_1' \times P_2'$. For, since Φ is closed, $F_{(1)}$ and $F_{(2)}$ are closed. The 1-forms $\tilde{\alpha}_i$ and $\tilde{\beta}_i$, $i = 1, 2$, will also be closed if the P_i' are totally geodesic. Although the metrics g_i of P_i' need not be Kaehlerian we do have the following result.

Theorem 11. *The direct product of the symplectic quartic structures* $P_i'(f_{(i)}, U_i, V_i, \tilde{\alpha}_i, \tilde{\beta}_i)$, $i = 1, 2$, *has a naturally induced almost complex structure given by* $\hat{J}$. *If the quartic structures are normal, then* $\hat{J}$ *is integrable, and conversely. If the* P_i' *are totally geodesic hypersurfaces and the ambient space is a cosymplectic manifold, then* $P_1' \times P_2'$ *is Kaehlerian.*

Corollary. *Let P be a totally geodesic hypersurface of a cosymplectic manifold with the induced symplectic quartic structure. Then the direct product of P with itself is Kaehlerian.*

Remarks. (a) Let $P(f, \eta^a, g)$, $a = 1, \cdots, 2\nu$, be a totally flat even dimensional pseudo-globally framed manifold. We have seen that $P(\tilde{f}, g)$ and $P(\hat{f}, g)$ are Kaehler manifolds. If P is compact, then its topology may be studied from several points of view, first as a compact Kaehler manifold and secondly by introducing a theory on $P(f, \eta^a, g)$ analogous to Weil's generalization of Hodge's theory of harmonic integrals on algebraic varieties. Whereas $\tilde{F}$ and $\hat{F}$ are the Kaehler 2-forms of $P(\tilde{f}, g)$ and $P(\hat{f}, g)$ respectively, F plays that role in the latter theory. Although f has maximal rank, (f, g) is not a Kaehler structure on P. However, if $\nu = n$, then (f, g) is Kaehlerian. This therefore yields a generalization of Kaehler geometry (see § 3).

(b) Let $P(f, E_a, \eta^a, \lambda)$ and $\bar{P}(\bar{f}, \bar{E}_a, \bar{\eta}^a, \bar{\lambda})$ be pseudo-globally framed spaces of the same dimension and rank. A diffeomorphism μ of P onto $\bar{P}$ is called an *isomorphism* of P onto $\bar{P}$ if

$$\mu_* \circ f = \bar{f} \circ \mu_*$$

and

$$\mu_* E_a = \bar{E}_a \, .$$

If $\bar{P} = P$ and $\bar{f} = f$, $\bar{E}_a = E_a$, $\bar{\eta}^a = \eta^a$ for all a, then μ is an *automorphism* of P. The set of all automorphisms of P clearly forms a group which we denote by $A(f, E_a, \eta^a)$. If $\mu \in A(f, E_a, \eta^a)$, then $\mu^* \eta^a = \eta^a$, hence $\bar{\lambda} \circ \mu = \lambda$. Moreover, $\mu_* \circ f_2 = f_2 \circ \mu_*$, where $f_2 = f - \lambda(\eta^{2i} \otimes E_{2i-1} - \eta^{2i-1} \otimes E_{2i})$. Thus $\mu \in A(f_2, E_a, \eta^a)$, from which we conclude that $\mu \in A(\tilde{f}, E_a, \eta^a) \cap A(\hat{f}, E_a, \eta^a)$.

Conversely, if $\mu \in A(\bar{f}, E_a, \eta^a) \cap A(\hat{f}, E_a, \eta^a)$, then $\mu \in A(f, E_a, \eta^a)$. If P is compact and $(f, E_a, \eta^a, \lambda)$ is normal, we conclude just as in [5] that the group of automorphisms of a pseudo-globally framed space is a Lie group.

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